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2×2 systems of conservation laws with $\mathbf{L}^\infty$ data

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ABSTRACT

Consider a hyperbolic system of conservation laws with genuinely nonlinear characteristic fields. We extend the classical Glimm–Lax (1970) result [13, Theorem 5.1] proving the existence of solutions for $\mathbf{L}^\infty$ initial datum, relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves.

1. Introduction

Consider the following nonlinear 2×2 system of conservation laws

$$\partial_t u + \partial_x \big[f(u) \big] = 0 \tag{1.1}$$

and the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \big[f(u) \big] = 0, \\ u(0,x) = \bar{u}(x). \end{cases} \tag{1.2}$$

Our aim is to extend the classical result [13, Theorem 5.1] relaxing the assumptions taken therein on the geometry of the shock–rarefaction curves. More precisely, as is well known, the assumptions in [13] ensure that the interaction of two shocks of the same family yields a shock of that family and a *rarefaction* of the other family. Here, no assumption whatsoever of this kind is assumed. Nevertheless,

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the result of Theorem 1.1 is the same of that in [13, Theorem 5.1], namely the existence of a weak entropy solution to (1.2) for all initial data with sufficiently small $\mathbf{L}^\infty$ norm.

On the flow f in (1.1) we assume the following Glimm–Lax condition, analogously to [13, formula (1.4)]:

(**GL**) $f : B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and with both characteristic fields genuinely nonlinear

where $B(0,r)$ is the ball of $\mathbb{R}^2$ with center 0 and radius r. The main result of this paper is the following:

Theorem 1.1. *Under the assumption* (**GL**), *there exists a sufficiently small* $\eta > 0$ *such that for every initial condition* $\bar{v} \in \mathbf{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^2)$ *with*:

$$\|\bar{v}\|_\infty \leqslant \eta \tag{1.3}$$

the Cauchy problem (1.2) *admits a weak entropy solution for all* $t \geqslant 0$.

The solution is constructed as limit of the ε-approximations v^ε constructed through the front tracking algorithm used in [6], suitably adapted to the present situation. First, as in [13], careful decay estimates on a trapezoid (see Fig. 2) allow to bound the positive variation and the $\mathbf{L}^\infty$ norm of v^ε on the upper side of the trapezoid. Under the further assumption that a suitable $\mathbf{L}^\infty$ estimate on v^ε holds, see condition (**A**), a technique based on the hyperbolic rescaling allows to extend the previous bound to any positive time. The approximate solutions can hence be defined globally in time.

A key point is now to provide estimates that allow to abandon condition (**A**). This is achieved through $\mathbf{L}^\infty$ estimates essentially based on the conservation form of (1.1) and on the previous results on the trapezoids. It is here that the integral estimates in Section 6 allow us to extend the result in [13].

As a byproduct, we also obtain Theorem 3.12, under the standard Lax condition

(**L**) $f : B(0,r) \to \mathbb{R}^2$, for a suitable $r > 0$, is smooth with $Df(0)$ strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate.

Indeed, Theorem 3.12 is an existence result valid for all initial data having small $\mathbf{L}^\infty$ norm and bounded, not necessarily small, total variation.

In this connection, we recall that in the case of systems with coinciding shock and rarefaction waves, the well posedness of (1.2) in $\mathbf{L}^\infty$ was proved in [4] under condition (**GL**), extending the previous results [3,8]. Another attempt towards an extension of Glimm–Lax result is in [9].

This paper is organized as follows. Section 2 is devoted to introduce the notation. Then, ε-approximate solutions are defined in Section 3 and suitable bounds are proved, in the case of bounded total variation. Section 4 uses the previous results to construct the ε-approximate solutions globally in time under the further assumption (**A**). This latter assumption is abandoned in Section 5, which relies on the integral estimates in Section 6. The more technical details are collected in the final Section 7.

2. Notations

As a general reference on the theory of conservation laws, we refer to [5,11]. Throughout, we let $B(u,r)$ be the open sphere in $\mathbb{R}^2$ centered at u with radius r.

Denote by $A(u)$ the 2×2 hyperbolic matrix $Df(u)$, by λ_1, λ_2 its eigenvalues and by l_1, l_2 (resp. r_1, r_2) its left (resp. right) eigenvectors, normalized so that

$$\|r_i(u)\| = 1, \qquad \langle l_j(u), r_i(u)\rangle = \begin{cases} 1 & j = i, \\ 0 & j \neq i, \end{cases} \quad i, j = 1, 2.$$

If the i-th characteristic field is genuinely nonlinear, we choose r_i oriented so that

$$D\lambda_i(u)r_i(u) \geqslant c > 0 \quad \text{for } i = 1, 2 \text{ and } u \in B(0, r) \tag{2.1}$$

for a suitable c. In the linearly degenerate case, we do not need to specify this orientation. By (**L**), $\sup_{B(0,r)} \lambda_1 < \inf_{B(0,r)} \lambda_2$.

By a linear change of coordinates, we can assume that $f(0) = 0$, $A(0) = \operatorname{diag}(\lambda_1(0), \lambda_2(0))$ and that $\lambda_1(0) = -1$, $\lambda_2(0) = 1$. We are thus led to assume that f can be written as follows:

$$\begin{aligned} f_1(u) &= -u_1 + \frac{1}{2}\alpha_{11}u_1^2 + \alpha_{12}u_1u_2 + \frac{1}{2}\alpha_{22}u_2^2 + \mathcal{O}(1)\|u\|^3, \\ f_2(u) &= u_2 + \frac{1}{2}\beta_{11}u_1^2 + \beta_{12}u_1u_2 + \frac{1}{2}\beta_{22}u_2^2 + \mathcal{O}(1)\|u\|^3 \end{aligned} \tag{2.2}$$

with $\alpha_{ij} := \frac{\partial^2 f_1}{\partial u_i \partial u_j}(0)$ and $\beta_{ij} := \frac{\partial^2 f_2}{\partial u_i \partial u_j}(0)$.

Following [5, formula (5.38)], introduce the Lax curves as the gluing of the shock and rarefaction curves:

$$L_i(u, \sigma) := \begin{cases} S_i(u, \sigma) & \sigma < 0, \\ R_i(u, \sigma) & \sigma \geqslant 0. \end{cases} \tag{2.3}$$

As in [5, formula (7.36)], call $E = E(u^-, u^+)$ the map giving the sizes of the waves in the solution to the Riemann problem for (1.1) with data u^- and u^+:

$$(\sigma_1, \sigma_2) = E\bigl(u^-, u^+\bigr) \quad \text{if and only if } u^+ = L_2\bigl(L_1\bigl(u^-, \sigma_1\bigr), \sigma_2\bigr).$$

Recall now the continuous version of the Glimm potentials, see [7, (1.14) and (1.15)] or [10, (4.2)–(4.4)]. Throughout, we assume that any $u \in \mathbf{BV}(\mathbb{R}; B(0, r))$ is right continuous. For a Borel $\Omega \subseteq \mathbb{R}$, define the wave measures μ_i for $i = 1, 2$, as

$$\mu_i(\Omega) := \int_\Omega \langle l_i(u), d\mu_c\rangle + \sum_{x \in \Omega} E_i\bigl(u(x-), u(x+)\bigr)$$

where μ_c is the continuous part of the weak derivative of u and, as usual, $\langle l_i(u), d\mu_c\rangle := \sum_{j=1}^n l_i^j(u)d\mu_c^j$. Below, we consider also the positive part of the signed measure μ_i, denoted by μ_i^+, and the positive total variation of the i-th component of u, denoted by $\mathrm{TV}^+(u_i)$. Then, let

$$\rho := |\mu_2| \otimes |\mu_1| + \sum_{i=1}^2 \bigl(\mu_i^- \otimes \mu_i^- + \mu_i^+ \otimes \mu_i^- + \mu_i^- \otimes \mu_i^+\bigr) \tag{2.4}$$

and, as in [2,5,7,10], set

$$\begin{aligned} Q(u) &:= \rho\bigl(\bigl\{(x, y) \in \mathbb{R}^2 \colon x < y\bigr\}\bigr), \\ V(u, I) &:= |\mu_1|(I) + |\mu_2|(I) \quad I \subseteq \mathbb{R} \text{ interval}, \\ \Upsilon(u) &:= V(u, \mathbb{R}) + Q(u) \end{aligned}$$

where $|\mu_i|$ is the total variation of measure μ, $V(u,\mathbb{R})$ is the *total strength of waves* in u and $Q(u)$ is the *interaction potential* of u. For a $u \in \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^2)$, define its total variation by:

$$\mathrm{TV}(u) := \sup\left\{ \sum_{i=1}^{2}\sum_{l=1}^{N}\bigl|u_i(x_l)-u_i(x_{l-1})\bigr|:\ \begin{array}{l} x_1,\dots,x_N \in \mathbb{R} \text{ with} \\ x_1 < \cdots < x_N \end{array} \right\}. \tag{2.5}$$

Obviously, the total variation and the functional $V(\cdot,\mathbb{R})$ are equivalent. In the following, for $L>0$, it will be useful also the notation:

$$\mathrm{TV}(u;L) := \sup_{x\in\mathbb{R}} \mathrm{TV}(u|_{[x,x+L]})$$

where $u|_{[x,x+L]}$ is the restriction of u to the interval $[x,x+L]$.

For a function $u:\mathbb{R}\to B(0,r)$, we use below the $\mathbf{L}^\infty$ norm

$$\|u\|_\infty := \operatorname*{supess}_{x\in\mathbb{R}}\bigl|u_1(x)\bigr| + \operatorname*{supess}_{x\in\mathbb{R}}\bigl|u_2(x)\bigr|.$$

Below, $\hat{\lambda}$ denotes an upper bound for the moduli of the characteristic speeds in $B(0,r)$, i.e.

$$\hat{\lambda} > \sup_{i=1,2;\ \|u\|\leqslant r}\bigl|\lambda_i(u)\bigr|. \tag{2.6}$$

3. Construction of solutions with bounded total variation and small $\mathbf{L}^\infty$ norm

In this section, we modify the wave front tracking algorithm in [6, Section 2] to construct a solution to (1.2) under the assumption that the initial datum has bounded total variation and small $\mathbf{L}^\infty$ norm. More precisely, let $\bar{u}$ belong to

$$\mathcal{D}(\eta,\bar{K}) := \bigl\{u \in \mathbf{L}^1_{\mathrm{loc}}\bigl(\mathbb{R}; B(0,\eta)\bigr):\ \mathrm{TV}(u) \leqslant \bar{K}\bigr\}, \tag{3.1}$$

where $\bar{K}$, η are positive constants.

Moreover, in the first two paragraphs below, it is not necessary to assume that both characteristic fields be genuinely nonlinear. The standard Lax [15, Section 9] condition (**L**) is sufficient.

3.1. The algorithm

Fix $\varepsilon>0$. Denote by v the Riemann coordinates of (1.1), see [11, Definition 7.3.2], and call $\mathcal{L}_i$, $\mathcal{R}_i$ and $\mathcal{S}_i$ the Lax, the rarefaction and the shock curves in the Riemann coordinates:

$$\mathcal{L}_i(v,\sigma) := \begin{cases} \mathcal{S}_i(v,\sigma) & \sigma<0, \\ \mathcal{R}_i(v,\sigma) & \sigma \geqslant 0. \end{cases} \tag{3.2}$$

In these variables, as in [6], we parametrize the rarefaction and the shock curves as follows:

$$\begin{aligned} &\mathcal{R}_1(v,\sigma) = (v_1+\sigma, v_2), \qquad \mathcal{S}_1(v,\sigma) = \bigl(v_1+\sigma, v_2+\hat{\psi}_2(v,\sigma)\sigma^3\bigr), \\ &\mathcal{R}_2(v,\sigma) = (v_1, v_2+\sigma), \qquad \mathcal{S}_2(v,\sigma) = \bigl(v_1+\hat{\psi}_1(v,\sigma)\sigma^3, v_2+\sigma\bigr) \end{aligned} \tag{3.3}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are suitable smooth functions of their arguments. First, the initial datum $\bar{v}$ is substituted by a piecewise constant $\bar{v}^\varepsilon$ such that:

$$\lim_{\varepsilon\to 0+}\bigl\|\bar{v}^\varepsilon - \bar{v}\bigr\|_{\mathbf{L}^1} = 0, \qquad \mathrm{TV}\bigl(\bar{v}^\varepsilon\bigr) \leqslant \mathrm{TV}(\bar{v}) \leqslant \bar{K}, \qquad \bigl\|\bar{v}^\varepsilon\bigr\|_\infty \leqslant \eta.$$

At each point of jump in $\bar{v}^\varepsilon$, the resulting Riemann problem is solved as in [6, Section 2]. Let $\varphi \in \mathbf{C}^\infty(\mathbb{R};\mathbb{R})$ be such that

$$\varphi(\sigma) = 1 \quad \text{for } \sigma \leqslant -2,$$
$$\varphi(\sigma) = 0 \quad \text{for } \sigma \geqslant -1,$$
$$\varphi'(\sigma) \in [-2, 0] \quad \text{for } \sigma \in [-2, -1]$$

and introduce the ε-approximate Lax curves

$$\mathcal{L}_i^\varepsilon(v, \sigma) = \varphi(\sigma/\sqrt{\varepsilon})\mathcal{S}_i(v, \sigma) + \big(1 - \varphi(\sigma/\sqrt{\varepsilon})\big)\mathcal{R}_i(v, \sigma) \quad \text{for } i = 1, 2.$$

An ε-solution to the Riemann problem for (1.1) with data v^-, v^+ is obtained gluing ε-rarefactions and ε-shocks. ε-rarefactions of the first, respectively second, family are substituted by rarefaction fans attaining values in $\varepsilon\mathbb{Z} \times \mathbb{R}$, respectively $\mathbb{R} \times \varepsilon\mathbb{Z}$, traveling with the characteristic speed of the state on the right of each wave. More precisely, similarly to [6, formulae (2.13)–(2.16)], in the case $i = 1$ of the first family, define $h, k \in \mathbb{Z}$ such that

$$h\varepsilon \leqslant v_1^- < (h+1)\varepsilon \quad \text{and} \quad k\varepsilon \leqslant \mathcal{R}_1\big(v^-, \sigma_1\big) < (k+1)\varepsilon.$$

Introducing $\omega_1^j = (j\varepsilon, v_2^-)$ for $j = h, \ldots, k$, define

$$v(t, x) := \begin{cases} v^- & x < \lambda_1(\omega_i^{h+1}) \cdot t, \\ \omega_1^j & \lambda_1(\omega_1^j)t \leqslant x < \lambda_1(\omega_1^{j+1})t \text{ for } h+1 \leqslant j \leqslant k-1, \\ \omega_1^k & \lambda_1(\omega_1^k)t \leqslant x < \lambda_1(\mathcal{R}_1(v^-, \sigma_1))t, \\ \mathcal{R}_1(v^-, \sigma_1) & \lambda_1(\mathcal{R}_1(v^-, \sigma_1))t \leqslant x. \end{cases} \tag{3.4}$$

The case of rarefaction waves of the second family is entirely similar.

A 1-shock with left state v^- and size σ_1, such that $\sigma_1 < -2\sqrt{\varepsilon}$, travels with the exact Rankine–Hugoniot speed $\lambda_1^s(v^-, \sigma_1)$. When $\sigma_1 > -2\sqrt{\varepsilon}$, we assign to this jump an interpolated speed λ_1^φ defined as an average between the exact Rankine–Hugoniot speed $\lambda_1^\varphi(v, \sigma)$ and an approximate characteristic speed, see [6, formulae (2.17), (2.18) and (2.19)]

$$\begin{aligned} \lambda_1^\varphi\big(v^-, \sigma_1\big) &:= \varphi(\sigma_1/\sqrt{\varepsilon})\lambda_1^s\big(v^-, \sigma_1\big) + \big(1 - \varphi(\sigma_1/\sqrt{\varepsilon})\big)\lambda_1^r\big(v^-, \sigma_1\big), \\ \lambda_1^r\big(v^-, \sigma_1\big) &:= \sum_j \frac{\operatorname{meas}([j\varepsilon, (j+1)\varepsilon] \cap [(\mathcal{S}_1(v^-, \sigma_1))_1, v_1^-])}{|\sigma_1|} \lambda_1\big(\omega_1^{j+1}\big). \end{aligned} \tag{3.5}$$

For every $\sigma_i < 0$, it holds

$$\lambda_i\big(\mathcal{S}_i\big(v^-, \sigma_i\big)\big) < \lambda_i^\varphi\big(v^-, \sigma_i\big) < \lambda_i\big(v^-\big). \tag{3.6}$$

2-shocks are treated similarly, we refer to [6, Section 2] for further details.

If the i-th characteristic family is linearly degenerate, the shock, the rarefaction and the ε-approximate Lax curves coincide. Moreover, the characteristic speed is constant along these curves, so that the interpolation (3.5) is trivial. Gluing the solutions to the Riemann problems at the points of jump in $\bar{v}^\varepsilon$ we obtain an ε-solution defined on a non-trivial time interval $[0, t_1]$, t_1 being the first time at which two or more waves interact. Any interaction yields a new Riemann problem, so that a piecewise constant ε-solution of the form

$$v^\varepsilon = \sum_\alpha v^\alpha \chi_{[x_\alpha, x_{\alpha+1}[} \quad \text{with } v^{\alpha+1} = \mathcal{L}_2^\varepsilon\big(\mathcal{L}_1^\varepsilon\big(v^\alpha, \sigma_{1,\alpha}\big), \sigma_{2,\alpha}\big) \tag{3.7}$$

is recursively extended in time. Hence, we obtain a sequence of ε-approximate solutions. Here, the meaning of by ε-approximate solutions is slightly different from that in [6, Definition 1], namely:

Definition 3.1. A piecewise constant function $v^\varepsilon = v^\varepsilon(t,x)$ is an *ε-approximate solution* if all its lines of discontinuities are ε-admissible wave fronts.

By an *ε-admissible wavefront* of the first family we mean a line $x = x(t)$ across which a function v^ε has a jump, say with $v^- = (v_1^-, v_2^-)$, $v^+ = (v_1^+, v_2^+)$, satisfying the following conditions:

- If $v_1^+ \geqslant v_1^-$, then $v_2^+ = v_2^-$ and

$$v_1^+ \leqslant v_1^- + \varepsilon, \qquad \dot{x} = \lambda_1\big(v^+\big). \tag{3.8}$$

- If $v_1^+ \leqslant v_1^-$, then $v^+ = \mathcal{L}_1^\varepsilon(v^-, \sigma_1)$ for some $\sigma_1 < 0$, $\dot{x}$ coincides with the speed λ_1^φ defined in [6, formula (2.19)] and satisfies

$$\lambda_1\big(v^+\big) < \dot{x} < \lambda_1\big(v^-\big). \tag{3.9}$$

The ε-admissible wave fronts of the second family are defined in an entirely similar way.

It may happen that three or more fronts interact at the same point. Due to the above algorithm, at least one of the interacting waves needs to be a shock. Then, similarly to [5, Remark 7.1] it is sufficient to slightly modify the speed of this incoming shock to avoid the multiple interaction. If this perturbation is small enough, the bound (3.9) is still true.

Above, we modified the wave propagation speed adopted in [6, Section 2]. The speeds defined therein have an essential role in the proof of the uniform Lipschitz dependence of the approximate solution from the initial datum. The present choice (3.4)–(3.5) is sufficient for [5, Propositions 2 and 3] to hold and allows for simpler proofs in the sequel.

3.2. Existence and properties of the approximate solutions

In this paragraph we show that the ε-approximate solutions constructed by the previous algorithm are well defined, see Theorem 3.10.

Throughout, by C we denote a positive constant dependent only on f and r as in (**L**).

The following lemma provides the standard interaction estimates.

Lemma 3.2. *There exists a positive C such that for any interaction resulting in the waves σ_1^+ and σ_2^+, the following estimates hold.*

1. *If the interacting waves are σ_1^- of the first family and σ_2^- of the second family,*

$$\big|\sigma_1^+ - \sigma_1^-\big| + \big|\sigma_2^+ - \sigma_2^-\big| = C\big|\sigma_1^-\sigma_2^-\big|\big(\big|\sigma_1^-\big| + \big|\sigma_2^-\big|\big).$$

2. *If the interacting waves σ' and σ'' both belong to the first family, we have*

$$\big|\sigma_1^+ - \big(\sigma' + \sigma''\big)\big| + \big|\sigma_2^+\big| = C\big|\sigma'\sigma''\big|\big(\big|\sigma'\big| + \big|\sigma''\big|\big).$$

3. *If the interacting waves σ' and σ'' both belong to the second family, we have*

$$\big|\sigma_1^+\big| + \big|\sigma_2^+ - \big(\sigma' + \sigma''\big)\big| = C\big|\sigma'\sigma''\big|\big(\big|\sigma'\big| + \big|\sigma''\big|\big).$$

The proof is in [6, Lemma 2 and Lemma 3].

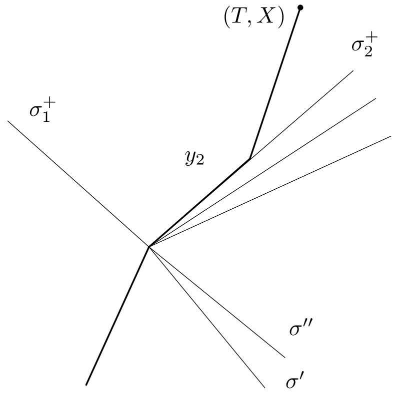

Fig. 1. Two 1-shock σ' and σ'' interact resulting in a 1-shock σ_1^+ and a 2-rarefaction σ_2^+. A 2-characteristic y_2 (thick line) is superimposed to the 2-rarefaction and passes through the interaction point.

Assume now that the ε-approximate solution v^ε is defined up to time $T > 0$. For $i = 1, 2$, $t \in [0, T]$ and $x \in \mathbb{R}$, introduce the quantities

$$\check{\lambda}_i(t, x) := \min\{\lambda_i(v^\varepsilon(t, x-)), \lambda_i(v^\varepsilon(t, x+))\},$$
$$\hat{\lambda}_i(t, x) := \max\{\lambda_i(v^\varepsilon(t, x-)), \lambda_i(v^\varepsilon(t, x+))\}.$$

For any $X \in \mathbb{R}$, the generalized i-th characteristic through (T, X) is an absolutely continuous solution $x(t)$ to the differential inclusion

$$\begin{cases} \dot{x} \in [\check{\lambda}_i(t, x), \hat{\lambda}_i(t, x)], \\ x(T) = X. \end{cases}$$

The *minimal* backward i-th characteristic through (T, X) is the generalized i-th characteristic such that, for $t \in [0, T]$,

$$y_i(t) := \min\{x(t)\colon x \text{ is a generalized } i\text{-th characteristic through } (T, X)\},$$

where we omit the dependence of $y_i(t)$ from (T, X). It is clear that $y_i(t)$ is well defined, for v^ε piecewise constant, see [1, Theorem 2, Chapter 2, § 1].

As a reference about minimal backward characteristics on exact solutions, see [11, Paragraph 10.3]. Backward characteristics on wave front tracking solutions were used, for instance, in [7, Section 4].

To estimate the norm $\|v^\varepsilon(T)\|_\infty$, for $T > 0$, we follow backward the i-coordinate v_i^ε along the minimal characteristic $y_i(t)$ through (T, X), for all $X \in \mathbb{R}$. Using the Lax inequality (3.6) and the choice adopted for the speed of rarefaction waves, we can conclude that y_i does not interact with any i-shock with size $\sigma < -\sqrt{\varepsilon}$, it can coincide on a non-trivial time interval with an i-wave with size $\sigma \geqslant -\sqrt{\varepsilon}$, it can cross a wave of the other family or pass through an interaction point where a rarefaction of its family arises, see Fig. 1.

In the lemma below, we denote $v(t^\pm, y_i(t^\pm)) := \lim_{\tau\to t^\pm} v(\tau, y_i(\tau))$.

Lemma 3.3. *Let $t > 0$ be such that $v_1(t^+, y_1(t^+)) \neq v_1(t^-, y_1(t^-))$. Then, either y_1 crosses a 2-wave σ_2, and*

$$\left|v_1^\varepsilon(t^+, y_1(t^+))\right| - \left|v_1^\varepsilon(t^-, y_1(t^-))\right| \leqslant C|\sigma_2|^3, \tag{3.10}$$

or y_1 passes through an interaction point between two waves σ', σ'' of the second family and

$$\left|v_1^\varepsilon\big(t^+, y_1(t^+)\big)\right| - \left|v_1^\varepsilon\big(t^-, y_1(t^-)\big)\right| \leqslant C\big(|\sigma'| + |\sigma''|\big)^3. \tag{3.11}$$

The proof directly follows from (3.3) and 3. in Lemma 3.2. An entirely analogous result holds along 2-characteristics.

The total size of the j-waves, with $j \neq i$, which may potentially interact with $y_i(t)$ after time t is given by the functionals

$$\tilde{Q}_1(t) := \sum_{\alpha:\, x_\alpha < y_1(t)} |\sigma_{2,\alpha}| \quad \text{and} \quad \tilde{Q}_2(t) := \sum_{\alpha: x_\alpha > y_2(t)} |\sigma_{1,\alpha}| \tag{3.12}$$

where we referred to the form (3.7) of v^ε. To estimate $\Delta\tilde{Q}_i(t)$, we analyze all the cases:

Lemma 3.4. *Let $i, j = 1, 2$ and $i \neq j$. Fix $t > 0$. If at time t there is*

1. *no interaction and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) = 0$;*
2. *no interaction and $y_i(t)$ crosses a j-wave σ_j, then $\Delta\tilde{Q}_i(t) = -|\sigma_j|$;*
3. *an interaction between σ' and σ'', and $y_i(t)$ does not cross any wave, then $\Delta\tilde{Q}_i(t) \leqslant C|\sigma'\sigma''|(|\sigma'| + |\sigma''|)$;*
4. *an interaction between the waves σ' and σ'', and $y_i(t)$ crosses a j-wave σ_j, then $\Delta\tilde{Q}_i(t) \leqslant C|\sigma'\sigma''| \times (|\sigma'| + |\sigma''|) - |\sigma_j|$;*
5. *an interaction between the j-waves σ' and σ'', and $y_i(t)$ crosses the interaction point, then $\Delta\tilde{Q}_i(t) \leqslant -|\sigma'| - |\sigma''|$.*

Proof. Points 1., 2. and 5. directly follow from the definition (3.12). Points 3. and 4. follow from Lemma 3.2 and (3.12). □

Now we also define, as usual, the *total strength of waves* and the *interaction potential*:

$$V\big(v^\varepsilon\big) := \sum_{i,\alpha} |\sigma_{i,\alpha}|, \qquad Q\big(v^\varepsilon\big) := \sum_{(\sigma_{i,\alpha}, \sigma_{j,\beta}) \in \mathcal{A}} |\sigma_{i,\alpha}\sigma_{j,\beta}|, \tag{3.13}$$

where $\mathcal{A}$ is the set of all couples of approaching wave-fronts, see [5, Paragraph 3, Section 7.3].

Proposition 3.5. *Fix a positive M'. Let the ε-approximate solution $v^\varepsilon = v^\varepsilon(t, x)$ be defined up to time $t > 0$. At time t an interaction between two waves σ' and σ'' takes place. If $\mathrm{TV}(v^\varepsilon(t^-)) < M'$ and $\|v^\varepsilon(t^-)\|_\infty$ is sufficiently small, then v^ε can be defined beyond time t and*

$$\Delta Q\big(v^\varepsilon(t)\big) \leqslant -\frac{|\sigma'\sigma''|}{2}.$$

Proof. Using Lemma 3.2 and (3.13), we have

$$\begin{aligned}\Delta Q\big(v^\varepsilon(t)\big) &\leqslant -\left|\sigma'\sigma''\right| + C\,\mathrm{TV}\big(v^\varepsilon(t^-)\big)\left|\sigma'\sigma''\right|\big(|\sigma'| + |\sigma''|\big) \\ &\leqslant \left|\sigma'\sigma''\right|\big(-1 + CM'\left\|v^\varepsilon(t^-)\right\|_\infty\big).\end{aligned}$$

Choosing $\|v^\varepsilon(t^-)\|_\infty < 1/(2CM')$, we obtain

$$\Delta Q\big(v^\varepsilon(t)\big) \leqslant -\frac{|\sigma'\sigma''|}{2}. \qquad \square$$

We introduce now the following two functionals:

$$\Upsilon^\varepsilon(t) := V\big(v^\varepsilon(t)\big) + KQ\big(v^\varepsilon(t)\big), \tag{3.14}$$

$$\Theta_i^\varepsilon(t) := \big(\big|v_i^\varepsilon\big(t, y_i(t)\big)\big| + \big\|\bar v^\varepsilon\big\|_\infty\big)e^{\tilde H\,\tilde Q_i(t)+HQ(v^\varepsilon(t))} \tag{3.15}$$

where $i = 1, 2$, $\tilde H$, H and K are positive constants to be precisely defined below. Note that the latter functional Θ_i^ε depends on the point (T, X). Nevertheless, the bound below holds uniformly in (T, X).

Proposition 3.6. *Fix positive M, M'. Choose an initial datum $\bar v^\varepsilon$ such that $\|\bar v^\varepsilon\|_\infty < \eta$. Assume that the ε-approximate solution $v^\varepsilon = v^\varepsilon(t,x)$ is defined up to time $t > 0$. If η is sufficiently small, $\mathrm{TV}(v^\varepsilon(t^-)) < M'$ and $\|v^\varepsilon(t-)\|_\infty < M\|\bar v^\varepsilon\|$, then there exist positive $\tilde H$, H and K of the order of $\|\bar v^\varepsilon\|_\infty$ such that*

$$\Delta\Upsilon^\varepsilon(t) \leqslant 0, \tag{3.16}$$

$$\Delta\Theta_i^\varepsilon(t) \leqslant 0 \quad \text{for } i = 1, 2. \tag{3.17}$$

In the proof below, we make use of the elementary inequality

$$e^A - e^B \leqslant (A - B)e^B.$$

Proof. First, we suppose that at time t there is no interaction and y_i crosses the wave σ_j. Obviously, $\Delta\Upsilon^\varepsilon = 0$ and $\|v^\varepsilon(t^+)\|_\infty = \|v^\varepsilon(t^-)\|_\infty$. Moreover:

$$\begin{aligned}
\Delta\Theta_i^\varepsilon(t) &= \big(\big|v_i^\varepsilon\big(t^+, y_i(t^+)\big)\big| + \big\|\bar v^\varepsilon\big\|_\infty\big)e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad - \big(\big|v_i^\varepsilon\big(t^-, y_i(t^-)\big)\big| + \big\|\bar v^\varepsilon\big\|_\infty\big)e^{\tilde H\tilde Q_i(t^-)+HQ(v^\varepsilon(t^-))} \\
&= \big(\big|v_i^\varepsilon\big(t^+, y_i(t^+)\big)\big| - \big|v_i^\varepsilon\big(t^-, y_i(t^-)\big)\big|\big)e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad + \big(\big|v_i^\varepsilon\big(t^-, y_i(t^-)\big)\big| + \big\|\bar v^\varepsilon\big\|_\infty\big)\big(e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} - e^{\tilde H\tilde Q_i(t^-)+HQ(v^\varepsilon(t^-))}\big) \\
&\leqslant C|\sigma_j|^3 e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} - \tilde H\big\|\bar v^\varepsilon\big\|_\infty|\sigma_j|e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\leqslant 0,
\end{aligned}$$

provided $\tilde H \geqslant CM^2\|\bar v^\varepsilon\|_\infty$.

Suppose now that at time t the waves σ' and σ'' interact and y_i does not pass through the interaction point. Hence, using Lemma 3.2 and the estimate of Proposition 3.5,

$$\Delta\Upsilon^\varepsilon(t) \leqslant C\big(\big|\sigma'\big| + \big|\sigma''\big|\big)\big|\sigma'\sigma''\big| - \frac{K}{2}\big|\sigma'\sigma''\big| \leqslant 0 \tag{3.18}$$

if $K \geqslant 2C(|\sigma'| + |\sigma''|)$. For the functional Θ_i^ε, we consider separately two cases. If $y_i(t)$ does not cross any wave at time t, we get:

$$\begin{aligned}
\Delta\Theta_i^\varepsilon(t) &\leqslant \big(\big|v_i^\varepsilon\big(t^-, y_i(t^-)\big)\big| + \big\|\bar v^\varepsilon\big\|_\infty\big)\big(e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} - e^{\tilde H\tilde Q_i(t^-)+HQ(v^\varepsilon(t^-))}\big) \\
&\leqslant \big\|\bar v^\varepsilon\big\|_\infty\left(C\tilde H\big(\big|\sigma'\big| + \big|\sigma''\big|\big) - \frac{H}{2}\right)\big|\sigma'\sigma''\big|e^{\tilde H\tilde Q_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\leqslant 0,
\end{aligned}$$

provided $H \geqslant 2C\tilde{H}(|\sigma'| + |\sigma''|)$. If $y_i(t)$ crosses a j-wave:

$$\begin{aligned}
\Delta\Theta_i^\varepsilon(t) &\leqslant \big(\big|v_i^\varepsilon\big(t^+, y_i\big(t^+\big)\big)\big| - \big|v_i^\varepsilon\big(t^-, y_i\big(t^-\big)\big)\big|\big)e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad + \big(\big|v_i^\varepsilon\big(t^-, y_i\big(t^-\big)\big)\big| + \|\bar{v}^\varepsilon\|_\infty\big)\big(e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+HQ(v^\varepsilon(t^-))}\big) \\
&\leqslant C|\sigma_j|^3 e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad + \|\bar{v}^\varepsilon\|_\infty\bigg(-\tilde{H}|\sigma_j| + C\tilde{H}\big(|\sigma'| + |\sigma''|\big)|\sigma'\sigma''| - \frac{H}{2}|\sigma'\sigma''|\bigg)e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\leqslant 0
\end{aligned}$$

provided $\tilde{H} > CM^2\|\bar{v}^\varepsilon\|_\infty$ and $H \geqslant 2C\tilde{H}(|\sigma'| + |\sigma''|)$.

Finally, we consider the case in which $y_i(t)$ is an interaction point where an i-rarefaction arises. Then, $\Delta\Upsilon(t) \leqslant 0$, as in (3.18), provided $K \geqslant 2C(|\sigma'| + |\sigma''|)$. Concerning $\Delta\Theta_i^\varepsilon(t)$, call σ', σ'' the sizes of the interacting j-waves.

$$\begin{aligned}
\Delta\Theta_i^\varepsilon(t) &\leqslant \big(\big|v_i^\varepsilon\big(t^+, y_i\big(t^+\big)\big)\big| - \big|v_i^\varepsilon\big(t^-, y_i\big(t^-\big)\big)\big|\big)e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad + \big(\big|v_i^\varepsilon\big(t^-, y_i\big(t^-\big)\big)\big| + \|\bar{v}^\varepsilon\|_\infty\big)\big(e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} - e^{\tilde{H}\tilde{Q}_i(t^-)+HQ(v^\varepsilon(t^-))}\big) \\
&\leqslant C\big(|\sigma'| + |\sigma''|\big)^3 e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\quad + \|\bar{v}^\varepsilon\|_\infty\bigg(-C\tilde{H}\big(|\sigma'| + |\sigma''|\big) - \frac{H}{2}|\sigma'\sigma''|\bigg)e^{\tilde{H}\tilde{Q}_i(t^+)+HQ(v^\varepsilon(t^+))} \\
&\leqslant 0
\end{aligned}$$

provided $\tilde{H} > 4CM^2\|\bar{v}^\varepsilon\|_\infty$ and $H \geqslant 2C\tilde{H}(|\sigma'| + |\sigma''|)$.

Note that all the constants $H, \tilde{H}$ and K are of the order of $\|\bar{v}^\varepsilon\|_\infty$. Indeed, a possible choice is

$$K = 4CM\|\bar{v}^\varepsilon\|_\infty, \qquad \tilde{H} = 4CM^2\|\bar{v}^\varepsilon\|_\infty \quad \text{and} \quad H = 16C^2M^3\|\bar{v}^\varepsilon\|_\infty.$$

Hence also the second part of the statement is proved. □

Proposition 3.7. *There exist positive M and C_2 such that, for all η, ε sufficiently small, if the ε-approximate solution $v^\varepsilon = v^\varepsilon(t, x)$ corresponding to the initial datum $\bar{v}^\varepsilon \in \mathcal{D}(\eta, \bar{K})$ is defined up to time T, then, for all $t \in [0, T]$,*

$$\mathrm{TV}\big(v^\varepsilon(t)\big) \leqslant C_2\bar{K} \quad \textit{and} \quad \big\|v^\varepsilon(t)\big\|_\infty \leqslant M\eta.$$

Proof. Let $t \in [0, T]$. To bound the $\mathbf{L}^\infty$ norm, for any $x \in \mathbb{R}$, first choose $\tilde{H} = 4CM^2\eta$ and $H = 16C^2M^3\eta^2$, as in Proposition 3.6. Then, recursively,

$$\begin{aligned}
\big\|v_i^\varepsilon(t)\big\| &\leqslant \Theta_i^\varepsilon(t) && \text{by (3.15)} \\
&\leqslant \Theta_i^\varepsilon(0) && \text{by Proposition 3.6} \\
&\leqslant 2\eta e^{4CM^2\eta(\tilde{Q}(0)+4CM\eta Q(0))} && \text{by (3.15)} \\
&\leqslant 2\eta e^{4CM^2\eta\bar{K}(1+4CM\bar{K}\eta)} \\
&\leqslant M\eta && \text{for } M = 2e^2,\ \eta < 1/\big(4CM^2\bar{K}\big)
\end{aligned}$$

for $i = 1, 2$. Taking the supremum with respect to x, we obtain the desired bound.

Similarly, to bound the total variation, apply recursively the previous results:

$$\begin{aligned}
\mathrm{TV}(v^\varepsilon(t)) &\leqslant C_1\Upsilon^\varepsilon(t) && \text{by (3.14)}\\
&\leqslant C_1\Upsilon^\varepsilon(0) && \text{by Proposition 3.6}\\
&\leqslant \bar K(1+4CM\bar K\eta) && \text{by (3.14)}\\
&\leqslant 2\bar K && \text{for } \eta < 1/(4CM^2\bar K)
\end{aligned}$$

completing the proof. □

Hence, by the Proposition 3.7, if $\bar v^\varepsilon \in \mathcal{D}(\eta, \bar K)$ and if the approximate solution v^ε can be constructed on some initial interval $[0, T]$, then $v^\varepsilon(t, \cdot) \in \mathcal{D}(M\eta, C_2\bar K)$ for all $t \in [0, T]$. In order to prove that v^ε can actually be defined for all $t > 0$, it remains to show that the total number of wave fronts and of points of interaction remains finite. For this aim, we use the next two propositions.

Proposition 3.8. *(See [6, Proposition 2].) Let $v^\varepsilon = v^\varepsilon(t, x)$ be an ε-approximate solution constructed by the previous algorithm, with $v^\varepsilon(t, \cdot) \in \mathcal{D}(M\eta, C_2\bar K)$ for all $t > 0$. Then, all of the shocks with size $\sigma < -\sqrt{\varepsilon}$ are located along a finite number of polygonal lines.*

Proposition 3.9. *(See [6, Proposition 3].) Let $v^\varepsilon = v^\varepsilon(t, x)$ be an ε-approximate solution constructed by the previous algorithm, with $v^\varepsilon(t, \cdot) \in \mathcal{D}(M\eta, \bar K)$ for all $t > 0$. Then, the set of all points where two fronts interact has no limit point in the (t, x)-plane.*

These two propositions are proved exactly as in [6]. The above results complete the proof of the following theorem.

Theorem 3.10. *Let (**L**) hold. Fix a positive $\bar K$. Then, there exist positive η and M such that for every initial condition $\bar v \in \mathcal{D}(\eta, \bar K)$ and for every sufficiently small $\varepsilon > 0$, the Cauchy problem (1.2) admits an ε-approximate solution $v^\varepsilon = v^\varepsilon(t, x)$ such that*

$$\left\| v^\varepsilon(t) \right\|_\infty \leqslant M\|\bar v\|_\infty. \tag{3.19}$$

Under condition (**GL**), we also have the following decay estimate.

Theorem 3.11. *Let (**GL**) hold. Fix a positive $\bar K$. Then, there exist positive η and $\mathcal{M}$ such that for every initial condition $\bar v \in \mathcal{D}(\eta, \bar K)$ and for every sufficiently small $\varepsilon > 0$, the ε-approximate solution $v^\varepsilon = v^\varepsilon(t, x)$ to the Cauchy problem (1.2) constructed in Theorem 3.10 satisfies for all $t > 0$, for all $a, b \in \mathbb{R}$ and for $i = 1, 2$:*

$$\mathrm{TV}^+\big(v_i^\varepsilon(t); [a, b]\big) \leqslant \frac{b-a}{ct} + \mathcal{M}\big(\|\bar v\|_\infty \mathrm{TV}\big(\bar v; [a - \hat\lambda t, b + \hat\lambda t]\big) + \varepsilon\big) \tag{3.20}$$

with c as in (2.1) and $\hat\lambda$ as in (2.6).

Notice that, in general, $\mathcal{M}$ depends on $\bar K$.

Proof. We recall the following decay estimate [5, formula (10.58)], see also [7]:

$$\mathrm{TV}^+\big(v_i^\varepsilon(t); [a, b]\big) \leqslant \frac{b-a}{ct} + C\big[Q\left(\bar v|_{[a-\hat\lambda t, b+\hat\lambda t]}\right) - Q\left(v^\varepsilon(t)|_{[a,b]}\right) + \varepsilon\big] \tag{3.21}$$

which holds under the present assumptions. Hence, we have

$$\mathrm{TV}^+\big(v_i^\varepsilon(t); [a,b]\big) \leqslant \frac{b-a}{ct} + C\,Q\,(\bar v|_{[a-\hat\lambda t, b+\hat\lambda t]}) + C\varepsilon.$$

Since the constants in Proposition 3.6 are of the order of $\|\bar v^\varepsilon\|_\infty$, then by inspection of the proof of [5, formula (10.58)], we note that the constant C in (3.21) can also be chosen of the order of $\|\bar v^\varepsilon\|_\infty$. Hence

$$\mathrm{TV}^+\big(v_i^\varepsilon(t); [a,b]\big) \leqslant \frac{b-a}{ct} + \mathcal{M}\big(\|\bar v\|_\infty \mathrm{TV}\big(\bar v; [a-\hat\lambda t, b+\hat\lambda t]\big) + \varepsilon\big)$$

completing the proof. $\square$

3.3. Existence of solutions

For the sake of completeness, we pass the ε-approximate solutions to the limit $\varepsilon \to 0$. This standard application of Helly compactness theorem yields a slight extension of the wave front tracking construction exhibited in [6]. Indeed, the mere existence of solutions to (1.2) is here obtained under the assumptions that the total variation of the initial datum be bounded.

Theorem 3.12. *Let* (**L**) *hold. Fix a positive $\bar K$. Then, there exist positive η, M such that for all $\bar u \in \mathcal{D}(\eta, \bar K)$, the Cauchy problem* (1.2) *admits a weak entropy solution, which is the limit of the wave front tracking approximate solutions constructed above and satisfying*

$$\big\|v(t)\big\|_\infty \leqslant M\|\bar v\|_\infty.$$

Moreover, if also (**GL**) *holds, then there exists a positive $\mathcal{M}$ such that for all $t > 0$, for all $a, b \in \mathbb{R}$ and for $i = 1, 2$,*

$$\mathrm{TV}^+\big(v_i(t); [a,b]\big) \leqslant \frac{b-a}{ct} + \mathcal{M}\|\bar v\|_\infty \mathrm{TV}\big(\bar v; [a-\hat\lambda t, b+\hat\lambda t]\big)$$

with c as in (2.1) *and $\hat\lambda$ as in* (2.6).

Thanks to the estimates proved above, the proof is standard and, hence, omitted.

4. Construction of a solution with small $\mathbf{L}^\infty$ norm

We now prove Theorem 1.1 in the case of initial data satisfying the stronger conditions

$$\bar v \in \mathbf{C}^1\big(\mathbb{R}; B(0,\eta)\big) \quad \text{with} \left\|\frac{d\bar v}{dx}\right\|_\infty \leqslant \mathcal{L}, \tag{4.1}$$

see [13, i), ii) and iii) in Section 5].

We are going to use an inductive method. Define, for $m = 0, 1, 2, \ldots$ and for every $L > 0$, the m-trapezoid by

$$\triangle_m := \big\{(t,x) \in [0,+\infty[\times\mathbb{R}\colon\ t \in [t_m, t_m + \Delta t_m] \text{ and } x \in \big[-2^m L + \hat\lambda(t-t_m), 2^m L - \hat\lambda(t-t_m)\big]\big\} \tag{4.2}$$

see Fig. 2, where:

$$t_m = \big(2^m - 1\big)L/2\hat\lambda \quad \text{and} \quad \Delta t_m = 2^{m-1}L/\hat\lambda. \tag{4.3}$$

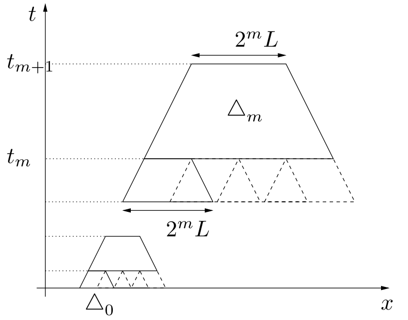

Fig. 2. Construction of the trapezoids.

The upper side of $\triangle_m$ measures $2^m L$ and the lower one $2^{m+1}L$. The upper bases of 4 trapezoids $\triangle_{m-1}$ cover the lower basis of $\triangle_m$. We denote by $\triangle_m(x)$ the translation of the m-trapezoid: $\triangle_m(x) := (0,x) + \triangle_m$. Correspondingly, we introduce the domains

$$\mathcal{D}_m\left(\delta, 20\frac{\hat{\lambda}}{c}\right) := \left\{ v \in \mathbf{L}^1_{\mathrm{loc}}\big(\mathbb{R}; B(0,\delta)\big)\colon \mathrm{TV}\big(v; 2^{m+1}L\big) \leqslant 20\frac{\hat{\lambda}}{c} \right\}. \tag{4.4}$$

4.1. *Construction in the* 0*-trapezoid*

In this paragraph we show that we are able to construct a solution in $\triangle_0(x)$, for all $x \in \mathbb{R}$. In fact, since the initial datum satisfies (4.1), we can always choose $L > 0$ such that

$$\mathrm{TV}(\bar{v}, 2L) \leqslant 20\hat{\lambda}/c. \tag{4.5}$$

Then, with reference to (4.4), we prove the following result.

Proposition 4.1. *Let* (**GL**) *and* (4.1) *hold. Then, there exist a sufficiently small* $\eta > 0$ *and positive* $M, \mathcal{M}$ *such that for every initial condition* $\bar{v} \in \mathcal{D}_0(\eta, 20\hat{\lambda}/c)$*, the Cauchy problem* (1.2) *admits a weak entropy solution* $v = v(t,x)$ *defined for all* $t \in [0, L/2\hat{\lambda}]$ *and*

$$\big\| v(t) \big\|_\infty \leqslant M \|\bar{v}\|_\infty,$$

$$\mathrm{TV}^+\big(v_i(t); 2(L - \hat{\lambda}t)\big) \leqslant \frac{2}{c}\frac{L - \hat{\lambda}t}{t} + \mathcal{M}\|\bar{v}\|_\infty \,\mathrm{TV}(\bar{v}; 2L).$$

The proof follows directly from Theorem 3.12.

4.2. *Construction in the m-trapezoid*

Now we prove that, if a solution v to (1.2) satisfies suitable conditions at time $t = t_m$, then this solution can be extended on all the interval $[t_m, t_{m+1}]$. We also provide suitable estimates for later use.

Proposition 4.2. *Let* (**GL**) *hold. Then, there exists a sufficiently small* $\eta > 0$ *and positive* $M, \mathcal{M}$ *such that if* $v(t_m) \in \mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$*, then the problem* (1.1) *with datum* $v(t_m)$ *admits a weak entropy solution* $v = v(t,x)$ *defined for* $t \in [t_m, t_{m+1}]$ *satisfying*

$$\left\| v(t) \right\|_\infty \leqslant M \left\| v(t_m) \right\|_\infty, \tag{4.6}$$

$$\mathrm{TV}^+\big(v_i(t); 2\big(2^m L - \hat{\lambda} t\big)\big) \leqslant \frac{2}{c} \frac{2^m L - \hat{\lambda} t}{t - t_m} + \mathcal{M} \left\| v(t_m) \right\|_\infty \mathrm{TV}\big(v(t_m); 2^{m+1} L\big). \tag{4.7}$$

Above, $\mathcal{D}_m(K\sqrt{\eta}, 20\hat{\lambda}/c)$ is defined in (4.4). The proof is entirely similar to that of Proposition 4.1.

4.3. *Existence of a global solution*

In this paragraph we assume the following a priory bound:

(**A**) Whenever it is possible to define up to time t_m a solution v to (1.2) with an initial datum satisfying (4.1), then there exists $K > 0$ such that, for all $m \in \mathbb{N}$, $\|v(t_m)\|_\infty \leqslant K\sqrt{\eta}$, where η is an upper bound for $\|\bar{v}\|_\infty$.

It is motivated by the recursive proof of Theorem 1.1 and by the following proposition.

Proposition 4.3. *Suppose there exists up to time t_m a weak entropy solution $v = v(t,x)$ to* (1.2) *with an initial datum satisfying* (4.1)*. Let* (**GL**), (4.5) *and* (**A**) *hold. Then, for all sufficiently small $\eta > 0$, if $\|\bar{v}\|_\infty \leqslant \eta$, for all $m \in \mathbb{N}$ we have the estimate*

$$\mathrm{TV}\big(v(t_m); 2^{m+1} L\big) \leqslant 20\frac{\hat{\lambda}}{c}.$$

Proof. Condition (4.5) immediately implies the desired bound for $m = 0$.

Let $m \geqslant 1$ and proceed by induction. Using the definition (4.2) of $\triangle_m(x)$ and the estimate (4.7), we get:

$$\begin{aligned}
\mathrm{TV}^+\big(v_i(t_m); 2^{m+1} L\big) &\leqslant 4\,\mathrm{TV}^+\big(v_i(t_m); 2^{m-1} L\big) \\
&\leqslant \frac{2^{m+1} L}{c(t_m - t_{m-1})} + 4\mathcal{M} \left\| v(t_{m-1}) \right\|_\infty \mathrm{TV}\big(v(t_{m-1}); 2^m L\big) \\
&\leqslant 8\frac{\hat{\lambda}}{c} + 4\mathcal{M} \left\| v(t_{m-1}) \right\|_\infty \mathrm{TV}\big(v(t_{m-1}); 2^m L\big).
\end{aligned}$$

Since $\mathrm{TV}(v) \leqslant (\mathrm{TV}^+(v_1) + \mathrm{TV}^+(v_2)) + 2\|v\|_\infty$, we obtain:

$$\mathrm{TV}\big(v(t_m); 2^{m+1} L\big) \leqslant 16\frac{\hat{\lambda}}{c} + 8\mathcal{M} \left\| v(t_{m-1}) \right\|_\infty \mathrm{TV}\big(v(t_{m-1}); 2^m L\big) + 2\left\| v(t_m) \right\|_\infty.$$

By (**A**) and choosing η small enough we get the thesis. □

Proof of Theorem 1.1 under condition (A). Assume first that the initial data satisfies (4.1). By an application of Proposition 4.1, we are able to construct a solution for all $t \in [0, L/2\hat{\lambda}]$. Now, assume that a solution exists up to time t_m, with $m \geqslant 1$. Then, by (**A**), we may apply Proposition 4.3 to obtain the TV bound at time t_m. Therefore, again thanks to (**A**), we apply Proposition 4.2 to extend the solution up to time t_{m+1}. The proof is thus obtained inductively.

Consider now a general initial datum satisfying only (1.3). As in [13, Section 5], we approximate the initial datum $\bar{v}$ by a sequence of mollified data $\bar{v}_n$ such that each $\bar{v}_n$ satisfies (4.1). So, we are able to construct a sequence of solutions v_n to (1.1) related to the initial data $\bar{v}_n$. Then by [11, Theorem 1.7.3] we can select a subsequence that converges to a limit v, which is a weak entropy solution to (1.2). □

5. The L^∞ estimate

The next step consists in proving that the a priori bound (**A**) is in fact a consequence of the other assumptions in Theorem 1.1 when the initial datum satisfies (4.1).

Proposition 5.1. *There exists a positive K such that for all initial datum $\bar{v}$ in* (1.2)*, satisfying* (1.3) *and for all $m \in \mathbb{N}$, on the solution $v = v(t,x)$ to* (1.2) *the following estimate holds*:

$$\left\| v(t_m) \right\|_\infty \leqslant K\sqrt{\eta},$$

where t_m is defined in (4.3).

Proof. For $m = 0$ the thesis holds, provided $K > \sqrt{\eta}$. Now, by induction, suppose that the theorem holds true up to $m-1$.

The lower basis of $\triangle_m$ is covered exactly by the upper basis of 4 $(m-1)$-trapezoids. Denote by T_{m-1} the union of these trapezoids. Then, divide T_{m-1} by horizontal segments $b^0_{m-1}, \dots, b^N_{m-1}$ into N sub-trapezoids, say $T^1_{m-1}, \dots, T^N_{m-1}$. Each sub-trapezoid T^j_{m-1} has height $h_N = 2^{m-2}L/(N\hat{\lambda})$, upper basis b^j_{m-1} and lower basis b^{j-1}_{m-1}, for $j = 1, \dots, N$. Obviously, b^0_{m-1} and b^N_{m-1} are the lower and upper basis of T_{m-1}.

At least one of these trapezoids, call it T^n_{m-1}, is such that

$$\begin{aligned}
&Q\left(v\big(t_{m-1} + (n-1)h_N\big)\big|_{b^{n-1}_{m-1}}\right) - Q\left(v(t_{m-1} + nh_N)|_{b^n_{m-1}}\right) \\
&\quad\leqslant \frac{1}{N}\left[Q\left(v(t_{m-1})|_{b^0_{m-1}}\right) - Q\left(v(t_m)|_{b^N_{m-1}}\right)\right] \\
&\quad\leqslant \frac{1}{N} Q\left(v(t_{m-1})|_{b^0_{m-1}}\right) \\
&\quad\leqslant \frac{1}{N}\left\| v(t_{m-1}) \right\|_\infty \mathrm{TV}\big(v(t_{m-1})\big) \\
&\quad\leqslant \frac{1}{N}\left\| v(t_{m-1}) \right\|_\infty \frac{20\hat{\lambda}}{c}
\end{aligned} \tag{5.1}$$

by Proposition 4.3. Now, fix (t,x) and (t,y) on b^n_{m-1} with $x < y$. Then, using together the decay estimate (3.21) on the region T^n_{m-1}, together with (5.1), we have:

$$v_i(t,y) \leqslant v_i(t,x) + \frac{N}{L}\frac{y-x}{2^{m-2}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty.$$

Integrate in y to obtain

$$\frac{1}{l}\int_x^{x+l} v_i(t,y)\,dy \leqslant v_i(t,x) + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty. \tag{5.2}$$

Similarly, integrating in x, we get

$$v_i(t,y) \leqslant \frac{1}{l}\int_{y-l}^{y} v_i(t,x)\,dx + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\| v(t_{m-1}) \right\|_\infty. \tag{5.3}$$

Using together (5.2) and (5.3), we obtain

$$\left|v_i(t,x)\right| \leqslant \frac{1}{l}\left|\int_{y-l}^{y} v_i(t,x)\,dx\right| + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_{\infty}. \tag{5.4}$$

At this point we consider three different cases, depending on which coefficients in (2.2) vanish. We defer the proofs of the corresponding integral estimates to Section 6.

1. $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$. Hence by Proposition 6.2,

$$\left|\int_{l} v_i(t,x)\,dx\right| \leqslant C'\eta\big(l + C''t\big) \quad \text{for } i = 1, 2. \tag{5.5}$$

   (Note that it is this case that covers the situation considered in [13].)
2. $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$. Then, using Proposition 6.3

$$\left|\int_{l} v_i(t,x)\,dx\right| \leqslant C'\eta\big(l + C''t\big) + C\left\|v(t)\right\|_{\infty}^{3} t \quad \text{for } i = 1, 2.$$

3. $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$). Hence, by an application of Proposition 6.4:

$$\left|\int_{l} v_i(t,x)\,dx\right| \leqslant C'\eta\big(l + C''t\big) + C\left\|v(t)\right\|_{\infty}^{3} t \quad \text{for } i = 1, 2.$$

Using the (worst) estimate of cases 2. and 3., we have

$$\left|v_i(t,x)\right| \leqslant C'\eta\left(1 + C''\frac{t}{l}\right) + C\left\|v(t)\right\|_{\infty}^{3}\frac{t}{l} + \frac{N}{L}\frac{l}{2^{m-1}}\frac{\hat{\lambda}}{c} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_{\infty}.$$

Setting $l/t = \sqrt{\eta + \|v(t)\|_{\infty}^{3}}$, using the fact that $t \leqslant t_m$ and the inductive assumption $\|v(t)\|_{\infty} \leqslant MK\sqrt{\eta}$, we have

$$\begin{aligned}
\left\|v(t)\right\|_{\infty} &\leqslant C(\eta + \sqrt{\eta}) + C\left\|v(t)\right\|_{\infty}^{3/2} + \frac{N}{c}\frac{\sqrt{\eta + \|v(t)\|_{\infty}^{3}}}{2^{m-1}}\frac{\hat{\lambda}}{L}t + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_{\infty} \\
&\leqslant C\sqrt{\eta} + \frac{CN}{c}\sqrt{\eta} + \frac{\mathcal{M}}{N}\frac{20\hat{\lambda}}{c}\left\|v(t_{m-1})\right\|_{\infty} \\
&\leqslant CN\sqrt{\eta} + \frac{C}{N}\left\|v(t_{m-1})\right\|_{\infty}.
\end{aligned}$$

Choosing $N = 4CM$ and $K = 4CMN$, by the inductive hypothesis, we get $\|v(t)\|_{\infty} \leqslant \frac{K}{2M}\sqrt{\eta}$. So, we can conclude by Proposition 4.1:

$$\left\|v(t_m)\right\|_{\infty} \leqslant M\left\|v(t)\right\|_{\infty} \leqslant \frac{K}{2}\sqrt{\eta}$$

completing the proof. Obviously, the proof is exactly the same if, instead of $\triangle_m$, we consider a generic trapezoid $\triangle_m(x)$ for some $x \in \mathbb{R}$. $\square$

Remark that in the previous proof, case 1 covers the situation treated in [13]. Indeed, in (5.5) the optimal choice for l/t is $l/t = \sqrt{\eta}$, exactly as in [13].

6. The integral estimate

Lemma 6.1. *Let $u = u(t,x)$ be the solution to (1.2) constructed in the previous sections, such that $\|u(t)\|_\infty \leqslant C\sqrt{\eta}$, with an initial data satisfying (1.3) and (4.1). If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ (respectively $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), then there exists an invariant region for the variable u_1 (respectively u_2). More precisely, there exists a positive constant $\mathcal{K}$ such that, for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, it holds:*

$$u_1(t,x) \geqslant -\mathcal{K}\eta, \quad \textit{respectively} \quad u_2(t,x) \geqslant -\mathcal{K}\eta.$$

Proof. At first we consider the ε-approximate solutions constructed above. Let v_1 and v_2 be the corresponding Riemann coordinates. The map $\mathcal{T} : v = (v_1, v_2) \mapsto u = (u_1, u_2)$ is smooth and maps the origin into the origin. So, using the hypothesis $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$, Lemma 7.2 implies that

$$\left[\dddot{\mathcal{S}}_2(v,\sigma) - \dddot{\mathcal{R}}_2(v,\sigma)\right]_1 = \left[\dddot{\mathcal{S}}_2(v,\sigma)\right]_1 \neq 0 \tag{6.1}$$

for v sufficiently small.

Let u^- and u^+ denote the left and the right states in a Riemann initial value problem, and let u^* denote the intermediate state, connected to u^- by a 1-wave and to u^+ by a 2-wave.

If $[\dddot{\mathcal{S}}_2(v,\sigma)]_1 \geqslant 0$ then we have that the Riemann invariant v_1^ε doesn't change along a right rarefaction and increases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \leqslant v_1^\varepsilon(u^+). \tag{6.2}$$

Obviously, this inequality holds also whenever the right shock has strength less then $2\sqrt{\varepsilon}$, in fact in this case we interpolate a rarefaction and an entropic shock. Using (6.2) and the fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \leqslant \eta$, we obtain $v_1^\varepsilon(t,x) \leqslant \eta$ for any $t > 0$. By a liner change of coordinates, we can assume that $\mathcal{T}_1(0,0) = 0$, $\frac{\partial \mathcal{T}_1}{\partial v_2}(0,0) = 0$ $\frac{\partial \mathcal{T}_1}{\partial v_1}(0,0) = -\mathcal{K}_1$, with $\mathcal{K}_1 > 0$. By this choice, it holds that $u_1^\varepsilon(t,x) = \mathcal{T}_1(v_1^\varepsilon(t,x), v_2^\varepsilon(t,x)) = -\mathcal{K}_1 v_1^\varepsilon(t,x) + \mathcal{K}_2(v_1^\varepsilon(t,x))^2 + \mathcal{K}_3 v_1^\varepsilon(t,x) v_2^\varepsilon(t,x) + \mathcal{K}_4(v_2^\varepsilon(t,x))^2$, where $\mathcal{K}_2$, $\mathcal{K}_3$ and $\mathcal{K}_4$ are the second derivatives of $\mathcal{T}_1$ computed in an intermediate point. Since $v_1^\varepsilon(t,x) < \eta$ and $\|v(t)\|_\infty \leqslant C\sqrt{\eta}$, we have $u_1^\varepsilon(t,x) \geqslant -\tilde{C}\mathcal{K}_1\eta - |\mathcal{K}_4|\eta$, for a suitable $\tilde{C} > 0$. Now, choosing $\mathcal{K} = \tilde{C}\mathcal{K}_1 + |\mathcal{K}_4|$, we obtain

$$u_1^\varepsilon(t,x) \geqslant -\mathcal{K}\eta.$$

Similarly, if $[\dddot{\mathcal{S}}_2(v,\sigma)]_1 \leqslant 0$, v_1^ε doesn't change along a right rarefaction and decreases along a right shock, i.e.

$$v_1^\varepsilon(u^*) \geqslant v_1^\varepsilon(u^+). \tag{6.3}$$

Now, using the fact that $v_1^\varepsilon(0,x) = \bar{v}_1^\varepsilon(x) \geqslant -\eta$ and (6.3), we get: $v_1^\varepsilon(t,x) \geqslant -\eta$ for any $t > 0$. As above, we can suppose that the map $\mathcal{T}_1$ is such that:

$$u_1^\varepsilon(t,x) \geqslant -\mathcal{K}\eta.$$

Clearly, the result still holds when we pass to the limit.

Similarly, if $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, it holds $u_2(t,x) \geqslant -\mathcal{K}\eta$. $\square$

Proposition 6.2. *Let $v = v(t,x)$ be the solution to* (1.2) *constructed in the previous sections, with an initial data satisfying* (1.3) *and* (4.1)*. If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$, then, for all segment l and for all $\bar{t} \geqslant 0$:*

$$\left| \int_l v_i(\bar{t}, x)\, dx \right| \leqslant C'\eta\big(l + C''\bar{t}\big). \tag{6.4}$$

Proof. By an application of Lemma 6.1, we get:

$$|u_1| \leqslant u_1 + 2\mathcal{K}\eta, \qquad |u_2| \leqslant u_2 + 2\mathcal{K}\eta. \tag{6.5}$$

Then, let us consider in the t,x plain the trapezoid with the lower basis l_0 equals to $[(0,x^l),(0,x^r)]$ and the upper basis l equals to $[(\bar{t}, x^l + \vartheta\bar{t}), (\bar{t}, x^r - \vartheta\bar{t})]$, where ϑ is positive. Then, using the Divergence Theorem

$$\begin{aligned}
\int_l \big[u_1(\bar{t},x) + u_2(\bar{t},x)\big]\, dx = {} & \int_{l_0} \big[u_1(0,x) + u_2(0,x)\big]\, dx \\
& - \int_{x^l}^{x^l+\vartheta\bar{t}} \left\{ \left[u_1\left(\frac{x - x^l}{\vartheta}, x\right) + u_2\left(\frac{x - x^l}{\vartheta}, x\right) \right] \right. \\
& \left. - \frac{1}{\vartheta}\left[f_1\left(u\left(\frac{x - x^l}{\vartheta}, x\right)\right) + f_2\left(u\left(\frac{x - x^l}{\vartheta}, x\right)\right) \right] \right\} dx \\
& - \int_{x^r-\vartheta\bar{t}}^{x^r} \left\{ \left[u_1\left(\frac{x^r - x}{\vartheta}, x\right) + u_2\left(\frac{x^r - x}{\vartheta}, x\right) \right] \right. \\
& \left. + \frac{1}{\vartheta}\left[f_1\left(\left(\frac{x^r - x}{\vartheta}, x\right)\right) + f_2\left(\left(\frac{x^r - x}{\vartheta}, x\right)\right) \right] \right\} dx.
\end{aligned} \tag{6.6}$$

Since f_1 and f_2 depend smoothly on u_1 and u_2 it holds that $|f_1| + |f_2| \leqslant C(|u_1| + |u_2|)$. Then, using this last estimate and (6.5) we get

$$\begin{aligned}
& \left[u_1\left(\frac{x - x^l}{\vartheta}, x\right) + u_2\left(\frac{x - x^l}{\vartheta}, x\right) \right] - \frac{1}{\vartheta}\left[f_1\left(u\left(\frac{x - x^l}{\vartheta}, x\right)\right) + f_2\left(u\left(\frac{x - x^l}{\vartheta}, x\right)\right) \right] \\
& \quad \geqslant \left(\left| u_1\left(\frac{x - x^l}{\vartheta}, x\right) \right| + \left| u_2\left(\frac{x - x^l}{\vartheta}, x\right) \right| \right) \left(1 - \frac{C}{\vartheta}\right) - 2\mathcal{K}\eta
\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}
& \left[u_1\left(\frac{x^r - x}{\vartheta}, x\right) + u_2\left(\frac{x^r - x}{\vartheta}, x\right) \right] + \frac{1}{\vartheta}\left[f_1\left(u\left(\frac{x^r - x}{\vartheta}, x\right)\right) + f_2\left(u\left(\frac{x^r - x}{\vartheta}, x\right)\right) \right] \\
& \quad \geqslant \left[u_1\left(\frac{x^r - x}{\vartheta}, x\right) + u_2\left(\frac{x^r - x}{\vartheta}, x\right) \right] - \frac{1}{\vartheta}\left[\left| f_1\left(u\left(\frac{x^r - x}{\vartheta}, x\right)\right) \right| + \left| f_2\left(u\left(\frac{x^r - x}{\vartheta}, x\right)\right) \right| \right] \\
& \quad \geqslant \left(\left| u_1\left(\frac{x^r - x}{\vartheta}, x\right) \right| + \left| u_2\left(\frac{x^r - x}{\vartheta}, x\right) \right| \right) \left(1 - \frac{C}{\vartheta}\right) - 2\mathcal{K}\eta.
\end{aligned} \tag{6.8}$$

We can choose $\vartheta = C$; now using (6.7) and (6.8) in the two last integrals on the right in (6.6) and (6.5) on the left, we get

$$\int_l \left[\left|u_1(\bar t, x)\right| + \left|u_2(\bar t, x)\right| - 2\mathcal{K}\eta\right] dx \leqslant \int_{l_0} \left[\left|u_1(0, x)\right| + \left|u_2(0, x)\right|\right] dx + 4\mathcal{K}C\bar t\eta$$

then

$$\int_l \left[\left|u_1(\bar t, x)\right| + \left|u_2(\bar t, x)\right|\right] dx \leqslant C'\eta\left(l + C''\bar t\right).$$

Since v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (6.4) is proved. □

Proposition 6.3. *Let $v = v(t, x)$ be the solution to* (1.2) *constructed in the previous sections, with an initial data satisfying* (1.3) *and* (4.1). *If $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, then, for all segment l and for all $\bar t \geqslant 0$:*

$$\left|\int_l v_i(\bar t, x)\, dx\right| \leqslant C'\eta\left(l + C''\bar t\right) + C\left\|v(\bar t)\right\|_\infty^3 \bar t. \tag{6.9}$$

Proof. Let us call l^- and l^+ the initial and the terminal point of l. For any curves $x^-(t)$ and $x^+(t)$ such that $x^-(\bar t) = l^-$ and $x^+(\bar t) = l^+$, by the Divergence Theorem, we get:

$$\begin{aligned}\int_l u_i(\bar t, x)\, dx = \int_{x^-(0)}^{x^+(0)} u_i(0, x)\, dx &+ \int_0^{\bar t} \left[f_i\left(u\left(t, x^-(t)\right)\right) - \dot x^-(t) u_i\left(t, x^-(t)\right)\right] dt \\ &+ \int_0^{\bar t} \left[-f_i\left(u\left(t, x^+(t)\right)\right) + \dot x^+(t) u_i\left(t, x^+(t)\right)\right] dt\end{aligned}$$

for $i = 1, 2$. Hence, to obtain

$$\left|\int_l u_i(\bar t, x)\, dx\right| \leqslant C'\eta\left(l + C''\bar t\right) + C\left\|u(\bar t)\right\|_\infty^3 \bar t \tag{6.10}$$

it is sufficiently to solve on $[0, \bar t]$ and out of shocks, up to terms of the order of $\|u(t)\|_\infty^2$, the ordinary differential equations:

$$\dot x^-(t) = \frac{f_i(u(t, x^-(t)))}{u_i(t, x^-(t))}, \qquad \dot x^+(t) = \frac{f_i(u(t, x^+(t)))}{u_i(t, x^+(t))}, \tag{6.11}$$

with the initial conditions $x^\pm(\bar t) = l^\pm$. By the hypothesis $\frac{\partial^2 f_i}{\partial u_j^2}(0) = 0$, (6.11) admit generalized solutions $x_i^-(t)$ and $x_i^+(t)$ in the sense of Filippov (see [12, Chapter 2, Section 4]). It may happen that their graph coincides with the support of shocks of the function u on sets of positive $\mathcal{H}^1$-measure. By Proposition 7.3, there exist two Lipschitz functions $\tilde x_i^\pm$ with $\tilde x_i^\pm(\bar t) = l^\pm$ and

$$\left\|\dot x_i^- - \dot{\tilde x}_i^-\right\|_\infty \leqslant \|u\|_\infty^2, \qquad \left\|\dot x_i^+ - \dot{\tilde x}_i^+\right\|_\infty \leqslant \|u\|_\infty^2$$

such that their graphs coincide with the shock of u on sets of zero $\mathcal{H}^1$-measure. Then, we have that (6.10) holds and, by the smoothness of v_1 and v_2, also the inequality (6.9) is proved. □

Proposition 6.4. *Let $v = v(t,x)$ be the solution to* (1.2) *constructed in the previous sections, with an initial data satisfying* (1.3) *and* (4.1)*. If $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$ (or $\frac{\partial^2 f_1}{\partial u_2^2}(0) = 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) \neq 0$), then, for all segment l and for all $\bar{t} \geqslant 0$:*

$$\left| \int_l v_i(\bar{t},x)\,dx \right| \leqslant C'\eta\big(l + C''\bar{t}\big) + C\big\|v(\bar{t})\big\|_\infty^3 \bar{t}. \tag{6.12}$$

Proof. Let us consider $\frac{\partial^2 f_1}{\partial u_2^2}(0) \neq 0$ and $\frac{\partial^2 f_2}{\partial u_1^2}(0) = 0$, in fact in the opposite case the proof is exactly the same. By an application of Lemma 6.1, we get:

$$|u_1| \leqslant u_1 + 2\mathcal{K}\eta. \tag{6.13}$$

Proceeding as in Proposition 6.2, we get:

$$\int_l \big[\big|u_1(\bar{t},x)\big| - 2\mathcal{K}\eta\big]\,dx = \int_{l_0} \big|u_1(0,x)\big|\,dx + 4\mathcal{K}C\bar{t}\eta$$

then:

$$\left| \int_l u_1(\bar{t},x)\,dx \right| \leqslant \int_l \big|u_1(\bar{t},x)\big|\,dx \leqslant C'\eta\big(l + C''\bar{t}\big). \tag{6.14}$$

For the variable u_2 we follow exactly the same strategy used in the Proposition 6.3, so that we obtain:

$$\int_l \big|u_2(\bar{t},x)\big|\,dx \leqslant C'\eta\big(l + C''\bar{t}\big) + C\big\|u(\bar{t})\big\|_\infty^3 \bar{t}. \tag{6.15}$$

Now, using together (6.14) and (6.15) and the fact that v_1 and v_2 are smooth functions of u_1 and u_2 also the inequality (6.12) is proved. □

7. Technical details

Lemma 7.1. *If f is as in* (2.2)*, then*

$$(Dr_2r_2)(0) = [-\alpha_{22}, 0]^T \quad \textit{and} \quad (Dr_1r_1)(0) = [-\beta_{11}, 0]^T. \tag{7.1}$$

Proof. Recall the definition of the resolvent: $R(\xi,u) := (A(u) - \xi I)^{-1}$ (see [14]). We have:

$$\begin{aligned} R(\xi,u) &= \big(A(0) + \big(A(u) - A(0)\big) - \xi I\big)^{-1} \\ &= \big(A(0) - \xi I\big)^{-1}\big(I + \big(A(u) - A(0)\big)\big(A(0) - \xi I\big)^{-1}\big)^{-1} \\ &= \big(A(0) - \xi I\big)^{-1} - \big(A(0) - \xi I\big)^{-1}\big(A(u) - A(0)\big)\big(A(0) - \xi I\big)^{-1} + \mathcal{O}\big(u^2\big). \end{aligned}$$

Choose a closed curve Γ such that $\lambda_2(u)$ is the unique eigenvalue inside it. The projection P_2 can then be computed as:

$$\begin{aligned}
P_2(u) &= -\frac{1}{2\pi i}\oint_\Gamma R(\xi,u)\,d\xi = -\frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} d\xi \\
&\quad + \frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(u)+1 & \frac{\partial f_1}{\partial u_2}(u) \\ \frac{\partial f_2}{\partial u_1}(u) & \frac{\partial f_2}{\partial u_2}(u)-1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\xi+1} & 0 \\ 0 & \frac{1}{1-\xi} \end{bmatrix} d\xi + \mathcal{O}(u^2) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2\pi i}\oint_\Gamma \begin{bmatrix} 0 & -\frac{\frac{\partial f_1}{\partial u_2}(u)}{(\xi+1)(1-\xi)} \\ -\frac{\frac{\partial f_2}{\partial u_1}(u)}{(\xi+1)(1-\xi)} & 0 \end{bmatrix} \\
&\quad + \mathcal{O}\left(\frac{1}{(1-\xi)^2}\right) + \mathcal{O}\left(\frac{1}{(\xi+1)^2}\right) d\xi + \mathcal{O}(u^2) \\
&= \begin{bmatrix} 0 & -\alpha_{12}u_1 - \alpha_{22}u_2 \\ -\beta_{11}u_1 - \beta_{12}u_2 & 1 \end{bmatrix} + \mathcal{O}(u^2).
\end{aligned}$$

Since $P_2(u) = r_2(u) \otimes l_2(u)$,

$$r_2(u) = [-\alpha_{12}u_1 - \alpha_{22}u_2, 1]^T + \mathcal{O}(1)\|u\|^2 \tag{7.2}$$

and

$$l_2(u) = [-\beta_{11}u_1 - \beta_{12}u_2, 1]^T + \mathcal{O}(1)\|u\|^2. \tag{7.3}$$

Finally $Dr_2(0) = \left[\begin{smallmatrix} -\alpha_{12} & -\alpha_{22} \\ 0 & 0 \end{smallmatrix}\right]$ and $(Dr_2 r_2)(0) = [-\alpha_{22}, 0]^T$.

To prove the second equation it is sufficient to repeat the previous arguments. □

Lemma 7.2. *If* $\frac{\partial^2 f_1}{\partial u_2^2}(0) = \alpha_{22} \neq 0$, $\frac{\partial^2 f_2}{\partial u_1^2}(0) = \beta_{11} \neq 0$ *and condition* (**GL**) *holds, then*

$$\begin{aligned}
\left[\dddot{S}_2(0,0) - \dddot{R}_2(0,0)\right]_1 &= \frac{1}{2}\frac{\langle (D\lambda_2\, r_2)(Dr_2\, r_2), r_1\rangle}{\lambda_2 - \lambda_1} \neq 0, \\
\left[\dddot{S}_1(0,0) - \dddot{R}_1(0,0)\right]_2 &= \frac{1}{2}\frac{\langle (D\lambda_1\, r_1)(Dr_1\, r_1), r_2\rangle}{\lambda_1 - \lambda_2} \neq 0.
\end{aligned}$$

Proof. Let us denote by $S_2(\sigma)$ and $R_2(\sigma)$ the shock and the rarefaction curve of the second family with starting point 0, by $A(\sigma)$ the Jacobian matrix $Df(S_2(\sigma))$, by $r_i(\sigma)$ ($l_i(\sigma)$) the right (left) eigenvector $r_i(S_2(\sigma))$ ($l_i(S_2(\sigma))$) and by Λ the Rankine–Hugoniot speed.

Differentiating three times the Rankine–Hugoniot conditions w.r.t. σ we obtain:

$$\ddot{A}\dot{S}_2 + 2\dot{A}\ddot{S}_2 + A\dddot{S}_2 = \dddot{\Lambda}S_2 + 3\dot{\Lambda}\ddot{S}_2 + 3\ddot{\Lambda}\dot{S}_2 + \Lambda\dddot{S}_2.$$

At $\sigma = 0$ it becomes

$$\ddot{A}r_2 + 2\dot{A}(Dr_2r_2) = \frac{3}{2}(D\lambda_2\, r_2)(Dr_2\, r_2) - A\dddot{S}_2 + 3\ddot{\Lambda}r_2 + \lambda_2\dddot{S}_2. \tag{7.4}$$

Differentiating twice w.r.t. σ the identity $Ar_2 = \lambda_2 r_2$ at $\sigma = 0$ we find

$$\begin{aligned}&\ddot{A}r_2 + 2\dot{A}(Dr_2r_2) + A(D^2r_2r_2)r_2 + ADr_2(Dr_2r_2)\\&\quad= \langle D^2\lambda_2 r_2, r_2\rangle r_2 + \langle D\lambda_2 Dr_2, r_2\rangle r_2 + 2(D\lambda_2 r_2)(Dr_2r_2) + \lambda_2\big(D^2r_2r_2\big)r_2 + \lambda_2 Dr_2(Dr_2r_2).\end{aligned}$$

Using (7.4) in the last equation:

$$\begin{aligned}&(A-\lambda_2\,\mathrm{Id})\big(D^2r_2r_2\big)r_2 + (A-\lambda_2\,\mathrm{Id})Dr_2(Dr_2r_2) - (A-\lambda_2\,\mathrm{Id})\dddot{S}_2 + 3\ddot{\Lambda}r_2\\&\quad= \langle D^2\lambda_2 r_2, r_2\rangle r_2 + D\lambda_2(Dr_2r_2)r_2 + \frac{1}{2}(D\lambda_2 r_2)(Dr_2r_2).\end{aligned}\tag{7.5}$$

Then, multiplying on the left by $l_2(0)$, it holds:

$$\ddot{\Lambda} = \frac{1}{3}D(D\lambda_2 r_2)r_2.\tag{7.6}$$

We can now substitute (7.6) in (7.5) and obtain

$$(\lambda_2\,\mathrm{Id}-A)\dddot{S}_2 = \frac{1}{2}(D\lambda_2 r_2)(Dr_2r_2) + (\lambda_2\,\mathrm{Id}-A)\big(D^2r_2r_2\big)r_2 + (\lambda_2\,\mathrm{Id}-A)Dr_2(Dr_2r_2).$$

Hence, multiplying on the left by $l_1(0) = [1,0] = r_1^T(0)$, we have that

$$\langle\dddot{S}_2, r_1\rangle = \frac{1}{2}\frac{\langle(D\lambda_2 r_2)(Dr_2r_2), r_1\rangle}{\lambda_2-\lambda_1} + \big\langle\big(D^2r_2r_2\big)r_2, r_1\big\rangle + \big\langle Dr_2(Dr_2r_2), r_1\big\rangle.$$

Now, since $\langle\dddot{R}_2, r^1\rangle = \langle(D^2r_2r_2)r_2, r_1\rangle + \langle Dr_2(Dr_2r_2), r_1\rangle$, using (7.1) and the genuine non-linearity, we can conclude that:

$$\langle\dddot{S}_2, r_1\rangle - \big\langle\dddot{R}_2, r^1\big\rangle = \frac{1}{2}\frac{\langle(D\lambda_2 r_2)(Dr_2r_2), r_1\rangle}{\lambda_2-\lambda_1} \neq 0.$$

The second part of the statement is proved repeating the same arguments. □

Proposition 7.3. *Let $u = u(t,x)$ be a weak entropy solution to* (1.2) *and denote by $\{y_m(t)\}_{m\in\mathbb{N}}$ the countable family of its shocks* (*see* [5, *Section* 10.3]). *Setting $L(T,X) := \{\varphi \in W^{1,\infty}[0,T]\colon\ \varphi(T) = X\}$ and $J := \bigcup_m \mathrm{graph}(y_m)$, we have that the set*

$$\mathcal{F} := \big\{\varphi \in L\colon\ \mathcal{H}^1\big(\mathrm{graph}(\varphi)\cap J\big) = 0\big\}$$

is dense in $L(T,X)$ endowed with the usual norm of $W^{1,\infty}$ (i.e. $\|\varphi\|_{W^{1,\infty}} := \|\varphi\|_\infty + \|\varphi'\|_\infty$).

Proof. L is complete, being a closed subset of a complete metric space. Observe that $\mathcal{F} = \bigcap_{m,n}\mathcal{F}_{n,m}$, where:

$$\mathcal{F}_{n,m} := \big\{\varphi \in L(T;X)\colon\ \mathcal{H}^1\big(\mathrm{graph}(\varphi)\cap\mathrm{graph}(y_m)\big) < 1/n\big\}.$$

By Baire Theorem, see [16, Proposition 3.5.4], it is sufficient to prove that each $\mathcal{F}_{n,m}$ is an open and dense subset of $L(T,X)$.

$\boldsymbol{\mathcal{F}_{n,m}}$ **is open:** Fix $\varphi \in \mathcal{F}_{n,m}$ and define

$$D_\varphi := \big\{ \big(t, y_m(t)\big) \in [0,T] \times \mathbb{R}\colon \varphi(t) = y_m(t) \big\},$$
$$D_\varphi^d := \big\{ \big(t, y_m(t)\big) \in [0,T] \times \mathbb{R}\colon \big|\varphi(t) - y_m(t)\big| \leqslant d \big\}.$$

For every $\varepsilon \in]0, 1/n - \mathcal{H}^1(D_\varphi)[$, there exists a $\delta > 0$ such that $\mathcal{H}^1(D_\varphi^\delta) = 1/n - \varepsilon$. Now, consider the open ball $\mathcal{B}(\varphi, \delta)$ in the space $(L(T,X), \|\cdot\|_{W^{1,\infty}})$. For every $\psi \in \mathcal{B}(\varphi,\delta)$, we have that $\psi(t) \neq y_m(t)$ whenever $(t, y_m(t)) \in \mathbb{R}^2 \setminus D_\varphi^\delta$. In fact, if $\psi(t) = y_m(t)$ with $(t, y_m(t)) \in \mathbb{R}^2 \setminus D_\varphi^\delta$, then $|\varphi(t) - \psi(t)| > \delta$ which is impossible since $\psi \in \mathcal{B}(\varphi,\delta)$. Hence, we obtain that $D_\psi \subseteq D_\varphi^\delta$, for all $\psi \in \mathcal{B}(\varphi,\delta)$, i.e. $\mathcal{B}(\varphi,\delta) \subset \mathcal{F}_{n,m}$. By the arbitrariness of φ, we conclude that $\mathcal{F}_{n,m}$ is open.

$\boldsymbol{\mathcal{F}_{n,m}}$ **is dense:** Choose a $\varphi \in L$. We show that φ can be arbitrarily approximated by functions in $\mathcal{F}_{n,m}$, hence we can assume that $\mathcal{H}^1(\mathrm{graph}(\varphi) \cap \mathrm{graph}(y_m)) \geqslant 1/n$. By [5, Theorem 10.4], $\varphi - y_m$ is Lipschitz on $[0,T]$. Then, call $\mathcal{C} = \{t \in [0,T]\colon\ \varphi(t) = y_m(t)\}$. $\mathcal{C}$ is closed and can be represented as $\mathcal{C} \subseteq \bigcup_{k=1}^N [a_k, b_k]$, for a suitable $N \geqslant 1$. Define, for instance, ψ as

$$\psi(t) := \varphi(t) + \delta^2 \sum_{k=1}^N e^{-1/((t-a_k)^2(b_k-t)^2)} \chi_{[a_k,b_k]}(t). \tag{7.7}$$

Clearly, $\psi \in \mathcal{F}_{n,m}$. Moreover $\|\varphi - \psi\|_{W^{1,\infty}} \leqslant \delta$, for δ small. Hence, $\psi \in \mathcal{B}(\varphi,\delta)$, proving the density of $\mathcal{F}_{n,m}$ in $L(T,X)$. $\square$

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References

[1] J.-P. Aubin, A. Cellina, Differential Inclusions, Set-Valued Maps and Viability Theory, Springer-Verlag, Berlin, 1984.
[2] P. Baiti, A. Bressan, The semigroup generated by a Temple class system with large data, Differential Integral Equations 10 (3) (1997) 401–418.
[3] P. Baiti, H.K. Jenssen, On the front-tracking algorithm, J. Math. Anal. Appl. 217 (2) (1998) 395–404.
[4] S. Bianchini, Stability of L^∞ solutions for hyperbolic systems with coinciding shocks and rarefactions, SIAM J. Math. Anal. 33 (4) (2001) 959–981 (electronic).
[5] A. Bressan, Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem, Oxford Lecture Ser. Math. Appl., vol. 20, Oxford University Press, Oxford, 2000.
[6] A. Bressan, R.M. Colombo, The semigroup generated by 2×2 conservation laws, Arch. Ration. Mech. Anal. 133 (1) (1995) 1–75.
[7] A. Bressan, R.M. Colombo, Decay of positive waves in nonlinear systems of conservation laws, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 26 (1) (1998) 133–160.
[8] A. Bressan, P. Goatin, Stability of L^∞ solutions of Temple class systems, Differential Integral Equations 13 (10–12) (2000) 1503–1528.
[9] C. Cheverry, Systèmes de lois de conservation et stabilité BV, Mém. Soc. Math. Fr. (N.S.) 75 (1998), vi+103.
[10] R.M. Colombo, G. Guerra, On the stability functional for conservation laws, Nonlinear Anal. 69 (5–6) (2008) 1581–1598.
[11] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, second ed., Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 325, Springer-Verlag, Berlin, 2005.
[12] A. Filippov, Differential Equations with Discontinuous Righthand Sides, first ed., Kluwer Academic Publisher, Dordrecht, The Netherlands, 1988.
[13] J. Glimm, P.D. Lax, Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws, Mem. Amer. Math. Soc., vol. 101, Amer. Math. Soc., Providence, RI, 1970.
[14] T. Kato, Perturbation Theory for Linear Operators, Classics Math., Springer-Verlag, Berlin, 1995, reprint of the 1980 edition.
[15] P.D. Lax, Hyperbolic systems of conservation laws. II, Comm. Pure Appl. Math. 10 (1957) 537–566.
[16] S.M. Srivastava, A Course on Borel Sets, Oxf. Grad. Texts Math., vol. 180, Springer-Verlag, New York, 1998.